NONSTEADY-STATE FLOW OF A VISCOPLASTIC MEDIUM IN A PLANE CHANNEL

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The problem of nonsteady-state plane flow is considered for a viscoplastic medium between parallel walls, under the action of an instantaneously applied pressure gradient which is constant in time. A nonlinear integrodifferential equation is obtained for the distribution of the tangential shear stresses in the regions investigated; the method of successive approximations is used to obtain the solution. The determination of the time during which the required interface reaches a specified position is reduced to the evaluation of a quadrature.

The problem of nonsteady-state flow of a viscoplastic medium at rest at a given instant in a plane channel under the action of an instantaneously applied pressure gradient which is time-constant has been the subject of a number of investigations. In [1], in order to determine the position in time of the required interface between the zones of viscous flow and quasisolid motion, Volterra's integral equations of the first species are obtained and the asymptotics of the solution for small time values are studied. The use of an integral two-sided Laplace transform with respect to a space variable permits the solution of this problem to be obtained in the form of a power series [2]; however, the solution obtained describes sufficiently well the asymptotic behavior of the interface between zones for small values of time but is less suitable for studying the behavior of its solution for large time values. The integral one-sided Laplace transform method with respect to a time variable, in the application to the problem of the nonsteady-state flow of a viscoplastic medium in a plane channel, reduces to some functional system of equations from which the position in time of the required zonal interface can be determined. The functional system obtained in this case is little suited to realistic calculations. The solution of the problem being considered by the integral Laplace transform method for the plane [4] and axisymmetrical [5] cases contains the principal error in the setting up of the boundary conditions, which is reviewed in detail in [3]. The analogous problem in [6] is solved by the Monte-Carlo statistical tests method. Despite the simplicity of carrying out the numerical procedure of the Monte-Carlo method, it is preferable to have an analytic expression for the position in time of the required zonal interface. The method of successive approximations, successfully applied in [7, 8] to the solution of Stefan's plane single-phase problems on freezing, is used here for constructing the solution of the problem concerning the nonsteady-state flow of a viscoplastic medium in a plane channel.

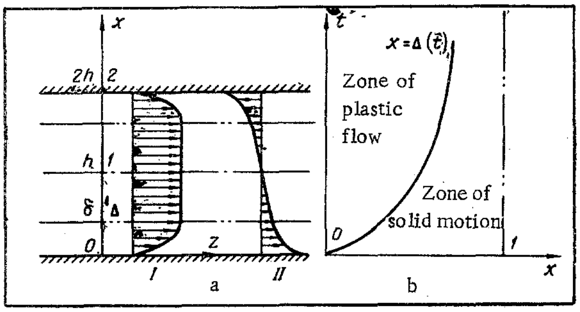

Fig. 1. Assumed flow pattern. I) Velocity distribution of medium; II) distribution of tangential shear stresses.

We shall consider the following problem. Suppose a viscoplastic medium with density ρ is at rest at $t < 0$ with a limiting shear stress τ_0 and dynamic coefficient of viscosity μ, flows in a plane channel of height 2h in the direction of the z-axis by the action of an instantaneously applied time-constant pressure against gradient $dp/dz < 0$. The assumed velocity distribution of the medium u and the tangential shear stresses τ for a certain instant $t > 0$, and also the

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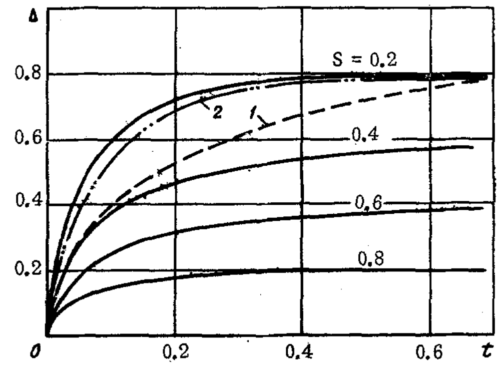

Fig. 2. Dependence of the position of the zone interface on time. 1) Result of first approximation; 2) result of second approximation.

arrangement of the system of coordinates are shown in Fig. 1a.

The special characteristic of one-dimensional flows of viscoplastic media is the possibility of formation of zones of quasisolid motion in which the absolute value of the tangential shear stresses is less than the limiting shear stress τ_0. The position of the required interfaces of the viscous zones and the zone of quasisolid motion for a certain instant $t > 0$ is shown in Fig. 1a by the dot-dash lines.

Because of the symmetry of the flow being considered, it is sufficient to limit construction of the solution to the lower half of the channel $x \in (0, h)$. The equation of motion of a viscoplastic medium in a plane channel and the rheological law connecting the tangential shear stresses with the rate of deformation of a solid medium have the form [9]

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial x} - \frac{dp}{\partial z}; \quad \frac{dp}{dz} = \text{const}, \tag{1}$$

$$\tau = \mu \frac{\partial u}{\partial x} + \tau_0 \operatorname{sign} \frac{\partial u}{\partial x}, \quad |\tau| \geqslant \tau_0, \tag{2}$$

$$\frac{\partial u}{\partial x} = 0, \quad |\tau| < \tau_0. \tag{3}$$

The condition for the existence of a quasisolid zone bounded by the surface $x = \delta(t)$ has the form

$$\tau(\delta(t), t) = \tau_0. \tag{4}$$

The wall of the channel $x = 0$ is assumed to be stationary, in consequence of which we have

$$u(0, t) = 0. \tag{5}$$

Movement of the quasisolid core of the flow $\delta(t) < x < h$ as a single entity is possible only by satisfying the following condition at the required boundary [3, 6]:

$$\frac{\partial \tau}{\partial x} = -\frac{\tau_0}{h - \delta(t)} \quad \text{for} \quad x = \delta(t). \tag{6}$$

Flow of the viscoplastic medium in the case being considered develops from a state of rest when the quasisolid zone has occupied the entire space between the channel walls and, therefore, we take as the initial condition for $\delta(t)$,

$$\delta(0) = 0. \tag{7}$$

Differentiating Eq. (1) with respect to the variable x and Eq. (2) with respect to the variable t, we obtain after simple transformations

$$\frac{\partial \tau}{\partial t} = \frac{\mu}{\rho} \cdot \frac{\partial^2 \tau}{\partial x^2}. \tag{8}$$

We note that the equation of motion in the form of Eq. (8) is valid only for the zone of viscous flow $0 < x < \delta(t)$, in which the rheological law in the form of Eq. (2) occurs. Using the condition of velocity continuity of the medium and the tangential shear stresses, by taking account of condition (5) and Eq. (1) we obtain

$$\frac{\partial \tau}{\partial x} = \frac{dp}{dz} \quad \text{for} \quad x = 0. \tag{9}$$

The problem of Eqs. (4) and (6)-(9) is written in τ-presentation [3, 6].

We introduce the dimensionless quantities:

$$T = (\tau - \tau_0)/\tau_{\text{char}} \quad \bar{x} = x/h; \quad \bar{t} = \frac{\mu}{\rho h^2} t;$$

$$\Delta = \delta/h; \quad s = \tau_0/\tau_{\text{char}} \text{ where } \tau_{\text{char}} = -h \frac{dp}{dz}.$$

Omitting the stroke over the dimensionless quantities $\bar{x}$ and $\bar{t}$, we rewrite the problem (4) and (6)-(9) in the following form:

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2} \text{ in } D=\{x,\ t: 0<x<\Delta(t),\ 0<t<M<\infty\}, \tag{10}$$

$$\frac{\partial T}{\partial x}=-1 \text{ for } x=0, \tag{11}$$

$$\frac{\partial T}{\partial x}=-\frac{s}{1-\Delta(t)} \text{ for } x=\Delta(t), \tag{12}$$

$$T=0 \text{ for } x=\Delta(t), \tag{13}$$

$$\Delta(0)=0. \tag{14}$$

The zone of plastic flow (region D) is shown at the phase plane (x, t) in Fig. 1b.

We integrate Eq. (10) with respect to the variable x in the limits from 0 to x and we use condition (11)

$$\frac{\partial T}{\partial x}=-1+\int_0^x \frac{\partial T}{\partial t}\,dx. \tag{15}$$

When $x = \Delta(t)$, Eq. (15) taking account of condition (12) assumes the form

$$\int_0^\Delta \frac{\partial T}{\partial t}\,dx=\frac{1-\Delta-s}{1-\Delta}. \tag{16}$$

It is easy to see that $\partial/\partial t = (\partial/\partial\Delta)\cdot(d\Delta/dt) = \dot{\Delta}(\partial/\partial\Delta)$, as a result of which we obtain from Eq. (16)

$$\dot{\Delta}=\frac{1-\Delta-s}{1-\Delta}\left(\int_0^\Delta \frac{\partial T}{\partial \Delta}\,dx\right)^{-1}. \tag{17}$$

As the RHS of Eq. (17) depends only on Δ, if we take into account the starting condition (14), we have

$$t=\int_0^\Delta \frac{(1-\Delta)\int_0^\Delta \frac{\partial T}{\partial \Delta}\,dx}{1-\Delta-s}\,d\Delta. \tag{18}$$

Expression (18) enables us to obtain $t = t(\Delta)$ if we plot the function $T = T(x, \Delta)$. We integrate Eq. (15) with respect to the variable x between the limits from x to Δ and we take into account the condition at the required boundary (13)

$$T(x,\ t)=\Delta-x-\int_x^\Delta\int_0^x \frac{\partial T}{\partial t}\,dx dx. \tag{19}$$

Taking relation (17) into account, it is easy to obtain

$$T(x,\ \Delta)=\Delta-x-\frac{1-\Delta-s}{1-\Delta}\cdot\frac{\int_x^\Delta\int_0^x \frac{\partial T}{\partial \Delta}\,dx dx}{\int_0^\Delta \frac{\partial T}{\partial \Delta}\,dx}. \tag{20}$$

The solution of Eq. (20) may be obtained by the method of successive approximations

$$T_{k+1}(x,\ \Delta)=\Delta-x-\frac{1-\Delta-s}{1-\Delta}\cdot\frac{\int_x^\Delta\int_0^x \frac{\partial T_k}{\partial \Delta}\,dx dx}{\int_0^\Delta \frac{\partial T_k}{\partial \Delta}\,dx}.$$

Taking as a first approximation

$$T_1=\Delta-x,$$

we obtain

$$T_2 = \Delta - x - \frac{1-\Delta-s}{1-\Delta}\cdot\frac{\Delta^2-x^2}{2\Delta}, \tag{22}$$

$$T_3 = \Delta - x - \frac{1-\Delta-s}{1-\Delta}\left[\left(1+\frac{s\Delta}{2(1-\Delta)^2}-\frac{1-\Delta-s}{2(1-\Delta)}\right)\frac{\Delta^2-x^2}{2}\right.$$
$$\left.-\frac{1}{6}\left(\frac{s}{\Delta(1-\Delta)^2}+\frac{1-\Delta-s}{\Delta^2(1-\Delta)}\right)\frac{\Delta^4-x^4}{4}\right]:$$
$$:\left[1+\frac{s\Delta}{3(1-\Delta)^2}-\frac{2}{3}\frac{1-\Delta-s}{1-\Delta}\right]. \tag{23}$$

Substituting in expression (18) the results of the successive approximation (21)-(23), we have respectively:

$$t_1 = \frac{1}{2}\Delta^2 - s\Delta + s(1-s)\ln\frac{1-s}{1-s-\Delta}, \tag{24}$$

$$t_2 = \frac{\Delta^2}{6} - \frac{2}{3}s\Delta + s(1-s)\frac{1-s}{1-s-\Delta} + \frac{(2-s)s}{6}\ln\frac{1-s-(2-s)\Delta+\Delta^2}{1-s} + \frac{2-2s+s^2}{6}\ln\frac{(1-\Delta)(1-s)}{1-\Delta-s}, \tag{25}$$

$$t_3 = \int_0^{\Delta}\frac{x(1-x)}{1-x-s}\left\{1+\frac{1}{A}\left[\frac{1-x-s}{1-x}\left(\frac{B'}{30}x^3+\frac{B}{6}x^2-\frac{1}{3}C'x-C\right)\right.\right.$$
$$\left.\left.+\frac{1}{3}\left(C-\frac{B}{10}x^2\right)\left(\frac{x(1-x-s)}{1-x}\cdot\frac{A'}{A}+\frac{x(1-x)+(1-2x)(1-x-s)}{(1-x)^2}\right)\right]\right\}dx, \tag{26}$$

where

$$A = \frac{1}{3}+\frac{s}{3}\cdot\frac{(2-x)}{(1-x)^2}; \quad B = \frac{(1-x)^2-s}{x^2(1-x)^2};$$

$$C = 1+\frac{s}{2}\cdot\frac{x}{(1-x)^2}-\frac{1-x-s}{2(1-x)};$$

the dash denotes the derivative of the corresponding function with respect to the variable x.

The nature of the convergence of the iteration process for $t = t(\Delta)$ is easily seen in Fig. 2, where the first, second, and third iterations are shown for values of the plasticity parameter $s = 0.2$ and also the results of the third approximation for certain values of the plasticity parameter. We note that $\Delta = \Delta(t)$ when $t > 0.6$ differs slightly from the value $\Delta_s = 1-s$, corresponding to completely steady-state flow.

NOTATION

t	is the time;
ρ	is the density;
τ_0	is the tangential shear stress;
μ	is the coefficient of dynamic viscosity;
h	is the half-width of channel;
z	is the longitudinal coordinate;
p	is the pressure;
u	is the velocity of medium;
x	is the transverse coordinate;
δ	is the function, describing the position of the boundary of the zone interface;
T	is the dimensionless tangential shear stress;
s	is the plasticity parameter;
Δ	is the dimensionless function, describing the position of the boundary of the zone interface;
D	is the region of plastic flow;
T_1, T_2, T_3, t_1, t_2, t_3	are the results of successive approximations;
A, B, C	are the auxiliary functions;
Δ_s	is the value of function Δ in the case of completely steady-state flow.

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